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Deductions from a simple climate model: Factors governing surface temperature and atmospheric thermal structure

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Abstract. Radiative equilibrium solutions are the starting point in our attempt to understand how the atmospheric composition governs the surface and atmospheric temperatures, and the greenhouse effect. The Schwarzschild analytical grey gas model (SGM) was the workhorse of such attempts. However, the solutions suffered from serious deficiencies when applied to Earth's atmosphere and were abandoned about 3 decades ago in favor of more sophisticated computer models. Here we show that simple heuristic modifications to the SGM resolve these deficiencies, at least the catastrophic ones. The modifications include the addition of a spectral window, as well as allowing the scale height of optical depth to be different from that of the atmospheric pressure. The modified SGM reveals the fundamental factors that govern the radiative equilibrium thermal structure. (1) The presence of a spectral window allows the temperature jump between the surface and the immediately overlying atmosphere to become small, irrespective of the magnitude of the surface temperature. (2) In an optically thick atmosphere (such as Earth and Venus), the surface temperature and the runaway greenhouse effect depend inversely (as the one-fourth power) on the Planck function-weighted width of the window and are independent of the optical depth and other atmospheric parameters. The vertical variation of temperature within the atmosphere, however, is determined by the vertical variation of optical depth. (3) The degree of convective instability of the radiative equilibrium thermal structure γ is proportional to the ratio H/H_τ, where H is the atmospheric scale height and H_τ is the scale height of the vertical variation of optical depth. Here, γ is the ratio of the radiative equilibrium lapse rate and the neutral lapse rate for dry convection; $\gamma > 1$ indicates that the profile is unstable to free convection. Thus $H_\tau < H$ emerges as one of the fundamental criteria for convective instability. These factors lead to several corollaries regarding the specific details of atmospheric thermal structure on Earth and Venus. For example, according to the modified SGM, the radiative equilibrium temperature profile is strongly superadiabatic in Earth's lower atmosphere because H_τ is dominated by the water vapor's scale height being much less (2 compared with 8 km) than the atmospheric scale height. In the case of Venus, the pressure broadening of the CO_2 rotational lines makes H_τ a factor of 2 smaller than H. Some of these results have been obtained using more detailed, multispectral radiative equilibrium models.

1. Introduction

Simple models of complex systems have great heuristic value, in that their results illustrate fundamental principles without being obscured by details. In particular, there exists a long history of simple climate models. Of these, radiative and radiative-convective equilibrium models have received great attention in the study of climate in general and Earth's climate in particular [*Manabe and Wetherald*, 1967; *Ramanathan and Coakley*, 1978]. One of the simplest radiative equilibrium models involves the assumption of a so-called grey atmosphere, where the absorption coefficient is assumed to be independent of wavelength. This was first discussed by *Schwarzschild* [1906] in the context of stellar interiors. The grey gas model was adapted to studies of the Earth by assuming the atmosphere to be transparent to solar radiation and grey for thermal radiation. We will refer to this latter class as semigrey models. A large number of publications on aspects of the semigrey atmosphere solutions exist in the meteorological literature, of which those by *Simpson* [1927, 1928] are of considerable relevance to the issues discussed in this paper. *Goody and Yung* [1989] provide an extensive discussion of semigrey atmosphere solutions and related topics pertaining to radiative and radiative-convective equilibrium.

In this paper, we extend the work of *Schwarzschild* [1906] and of *Goody and Yung* [1989]. We demonstrate that several interesting deductions about the stability of the atmospheric and surface temperatures can be obtained by minor modifications to the Schwarzschild equations. These modifications include (1) adding a spectrally transparent region (window) to the Schwarzschild model, (2) allowing the scale height of the primary absorbing gas to be different from that of the primary atmospheric constituents, and (3) accounting for the effects of strong, nonoverlapping lines within the context of

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the semigrey optical depth definition. The motivation for these modifications is obvious if we consider the Earth's atmosphere. It does not follow wavelength-independent absorption in the longwave, but has a broad window in the 8- to 12-μm region. The primary absorbing and emitting gases are H_2O and CO_2, while the gases that contribute most to the mass of the atmosphere are N_2 and O_2. Within the lower atmosphere, the centers of the H_2O and CO_2 rotational lines are saturated, such that the transfer of radiation between levels occurs through line wings. The line center optical depth is independent of the total atmospheric pressure, whereas the line wing optical depth increases with increasing pressure.

Surprisingly, although we invoke heuristic arguments to include these effects, the modified Schwarzschild model is able to simulate some of the characteristics of more sophisticated radiative equilibrium solutions and yields valuable insights into the factors governing the thermal structure of planetary atmospheres. Although our primary focus is the Earth's atmosphere, we will point out applications to other planets where applicable.

2. Schwarzschild's Grey Model

If we assume radiative equilibrium and a planetary atmosphere that is transparent to solar radiation, the two-stream Schwarzschild equations for the upwelling and downwelling flux of thermal radiation (F_s) are [*Schwarzschild*, 1906; *Houghton*, 1986]

$$\frac{dF_s^+}{d\tau} = D(F_s^+ - \pi B) \tag{1a}$$

$$\frac{dF_s^-}{d\tau} = -D(F_s^- - \pi B) \tag{1b}$$

where positive and negative signs indicate upwelling (toward the top of the atmosphere) and downwelling (toward the surface), respectively, and D is a constant resulting from the integration of the radiance over a hemisphere, usually taken to be 3/2 [*Goody and Young*, 1989]. The optical depth τ and the Planck function B are given by

$$\tau(z) = K \int_z^\infty p \; dz \tag{2a}$$

$$\tau^* = K \int_0^\infty p \; dz \tag{2b}$$

$$\pi B(\tau) = \sigma T(\tau)^4 \tag{2c}$$

where K is the pressure absorption coefficient, p is the absorber partial pressure, and σ is the Stefan-Boltzmann constant. K is not a function of wavelength. The subscript s in F_s simply indicates that the flux is as calculated from the Schwarzschild equations.

The requirement of radiative equilibrium everywhere in the atmosphere is satisfied by the following conditions:

$$\frac{dF_s}{d\tau} = 0 \tag{3a}$$

$$F_s = F_s^+ - F_s^- = f_0 \qquad \forall\, \tau \tag{3b}$$

where F_s is the net flux and f_0 is the net (down minus up) incoming solar radiation at the surface. For global mean conditions, f_0 is given by

$$f_0 = \frac{S}{4}(1 - \text{alb}) \tag{4}$$

where S is the solar constant and alb is the global mean planetary albedo. The solutions presented in this paper are not restricted to the global mean case but are valid for any f_0. The two required boundary conditions are

$$F_s(0) = F_s^+(0) = f_0 \tag{5a}$$

$$F_s^+(\tau^*) = \pi B_g = f_0 + F_s^-(\tau^*) \tag{5b}$$

The solution of (1), subject to (3) and (5), is

$$\sigma T^4 = f_0 \frac{(1 + D\tau)}{2} \tag{6a}$$

$$\sigma T_g^4 = f_0 \frac{(2 + D\tau^*)}{2} \tag{6b}$$

where T_g is the temperature of the surface.

There are two properties of (6a) and (6b) that we wish to examine further: temperature discontinuity and runaway greenhouse effect.

Temperature discontinuity. Equations (6a) and (6b) predict a temperature jump at the surface:

$$\sigma T_g^4 - \sigma T(\tau^*)^4 = \frac{f_0}{2} \tag{7}$$

This is a characteristic feature of semigrey atmosphere solutions. At large optical depths and, consequently, very high temperatures, this discontinuity becomes small relative to the emission near the surface.

Runaway greenhouse effect. From (6b), the surface temperature can increase without bounds as τ^* increases. *Simpson* [1927] identified the potential runaway greenhouse problem in the consideration of a semigrey atmosphere for Earth by showing the inability of an atmosphere in which water vapor is the primary absorber to respond to changes in the insolation. He went on to propose that the water vapor window prevented such a runaway condition [*Simpson*, 1928]. In most planetary atmospheres, there are always spectral regions that are not opaque, while other regions may be optically thick. Intuitively, we would expect that the spectral gappiness will limit the maximum value of T_g, since radiation can escape to space through the gaps. There is, however, another limiting factor for the semigrey atmosphere case. When T_g becomes very large, say, of the order of 1000 K or more, the spectrum of emission shifts to shorter wavelengths, and thus the assumption of grey optical depth for emission is inconsistent with the assumption of transparency to solar radiation. In such cases, (6a) and (6b) should be modified to those given in the appendix.

3. Modified Schwarzschild's Grey Model

3.1. Semigrey Model With One Spectral Window

The first modification is a simple extension of the Schwarzschild equations in which we add a spectral window in the thermal spectrum. We assume a fraction β of the total thermal spectrum is transparent to thermal radiation. That is, in this window region, the surface radiates directly to space as a blackbody, while elsewhere in the spectrum the flux satisfies the Schwarzschild equations. We assume the following.

1. The fraction β can be obtained by weighting with the Planck function:

$$\beta = \frac{\int_{\lambda_1}^{\lambda_2} B_\lambda(T)\, dT}{\int_0^\infty B_\lambda(T)\, dT} \tag{8}$$

where λ_1 and λ_2 determine the width of the window.

2. The fraction β is independent of temperature.

We can write the equation for the total upwelling and downwelling flux in the following manner:

$$F^+ = (1-\beta)F_s^+ + \beta\pi B_g \tag{9a}$$

$$F^- = (1-\beta)F_s^- \tag{9b}$$

where F_s is again the flux from (1a) and (1b) and πB_g is the surface emission. The requirement of radiative equilibrium everywhere in the atmosphere is now satisfied by the following condition:

$$F = F^+ - F^- = f_0 \qquad \forall\, \tau \tag{10}$$

where F is the net flux and f_0 is, again, the incoming solar radiation.

In a similar manner to the preceding section, we solve for the temperatures of the surface and the atmosphere:

$$\sigma T^4 = f_0 \frac{(1+D\tau)}{(2+\beta D\tau^*)} \tag{11a}$$

$$\sigma T_g^4 = f_0 \frac{(2+D\tau^*)}{(2+\beta D\tau^*)} \tag{11b}$$

With this modification, the properties of the Schwarzschild grey model (SGM) change as follows.

Temperature discontinuity.

$$\sigma T_g^4 - \sigma T(\tau^*)^4 = f_0 \frac{1}{(2+\beta D\tau^*)} \tag{12}$$

Thus, as opposed to the SGM, the modified equation predicts that the discontinuity would decrease as $1/\beta$, as well as $1/\tau^*$. In nongrey solutions that simulate the Earth's atmosphere, the temperature discontinuity is generally small, though not zero, and decreases as optical depth increases. It is the presence of spectral windows that allows the surface to rid itself of excess radiation in an efficient manner, which minimizes the discontinuity in the modified SGM as well as in sophisticated numerical models [e.g., *Manabe and Strickler*, 1964].

Runaway greenhouse effect. If we let $\tau^* \to \infty$ in (11b), we have

$$\sigma T_g^4 \to \frac{f_0}{\beta} \tag{13}$$

As $\tau^* \to \infty$, the surface temperature asymptotes to a constant value. For the Earth's atmosphere, $\beta \approx 0.3$ and $T_g \approx 340$ K, in reasonable agreement with radiative equilibrium surface temperature [*Manabe and Wetherald*, 1967].

Once the optical depth of the grey portion of the atmosphere is sufficiently large, the surface temperature depends only on the size of the window. For the surface temperature to increase, the size of the window must decrease. In other words, the modified SGM predicts that a runaway greenhouse effect is not possible in an atmosphere with a spectral window, no matter how small that window is. Another way to summarize this result, which is of particular relevance to the Earth's atmosphere, is as follows: in an optically thick atmosphere, the surface temperature is limited by the size of the spectral window.

This behavior is also apparent in the expression for the atmospheric greenhouse effect and the downward flux at the surface, which are identical in the semigrey model:

$$G_a = F^+(\tau^*) - F^+(0) = F^-(\tau^*) = f_0 \frac{D\tau^*(1-\beta)}{(2+\beta D\tau^*)} \tag{14}$$

The energy trapped in the column is equal to the downward emission. In order for the surface temperature to increase without bounds as τ^* increases, β must approach zero (i.e., the atmosphere approaches the limit given by the solutions to the SGM).

In the Earth's atmosphere, the 8- to 12-μm region of the longwave spectrum acts as a window, with the weak water vapor continuum absorption and the 9.6-μm band of ozone providing opacity. In addition, clouds can absorb significantly in this region, though we have not explicitly considered their impact in this paper. One way to decrease the size of Earth's window is to increase the water vapor concentration of the atmosphere, thereby increasing the continuum absorption. This can provide a powerful climate feedback, since thermodynamics dictates an increase in H_2O saturation vapor pressure with an increase in temperature. In addition, β can change because the wavelength of peak emission depends on temperature through Wien's law. If the Earth's temperature increased or decreased dramatically, the fraction of the blackbody curve present in the 8- to 12-μm region of the spectrum would change significantly. Since β represents the fraction of the integrated blackbody emission that is contained in the window region, as the emission peak shifts, β increases or decreases, even without a change in atmospheric composition. This provides an additional feedback, which is positive or negative, depending on the position of the window relative to the peak of the Planck curve for the given temperature.

3.2. Effects of Absorber Scale Height

One of the fundamental deductions from radiative equilibrium solutions concerns the stability of the vertical temperature structure to convection. If the radiative equilibrium temperature lapse rate ($-dT/dz$) is steeper than the neutral lapse rate for convection, then the radiative equilibrium atmosphere is considered to be absolutely unstable. Radiative equilibrium then yields to radiative-convective equilibrium as the dynamics of convection come into play. If $\Gamma_R = -dT/dz$ is the radiative equilibrium lapse rate and Γ_D is the neutral convection lapse rate, then

$$\gamma = \frac{\Gamma_R}{\Gamma_D} \geq 1$$

is absolutely unstable, and

$$\gamma = \frac{\Gamma_R}{\Gamma_D} < 1$$

is stable. Here, γ is the degree of convective instability of the radiative equilibrium thermal structure. Within about 10% accuracy, we can assume (following *Schwarzschild* [1906]) that partial pressure p for a uniformly mixed gas varies as

$$p = p^* e^{-z/H} \quad (15a)$$

where p^* is pressure at the surface. We can then write τ as

$$\tau = \tau^* e^{-z/H} \quad (15b)$$

$$H = \frac{RT}{g} \quad (15c)$$

R is the atmospheric gas constant and H is the atmospheric scale height. A uniformly mixed absorber has the same scale height as the atmosphere as a whole. We then obtain Γ_R from (6a) (or (11a); the solutions are identical due to the substitution of $H = RT(\tau)/g$) by differentiating the expression for $T(\tau(z))$ with respect to altitude. Therefore

$$\frac{\Gamma_R}{\Gamma_D} = \frac{C_p}{4R}\left(\frac{D\tau}{1 + D\tau}\right) \quad (16)$$

where we have used $\Gamma_D = g/C_p$. There are two aspects to this solution:

1. As τ decreases, Γ_R/Γ_D decreases, leading to the stable stratosphere.

2. Γ_R/Γ_D increases with τ and for $\tau \gg 1$, $\Gamma_R/\Gamma_D \approx C_p/4R$. Note that since $C_p/R \approx 3/2$ + number of atoms in a molecule, $C_p/R \approx 5/2$ for a monatomic gas, and $C_p/R \approx 7/2$ for a diatomic gas. Equation (16) yields

$$\frac{\Gamma_R}{\Gamma_D} \approx \frac{5}{8}$$

for a monatomic atmosphere and

$$\frac{\Gamma_R}{\Gamma_D} \approx \frac{7}{8}$$

for a diatomic atmosphere.

This leads to the conclusion drawn by *Schwarzschild* [1906], that a primarily monatomic or diatomic atmosphere cannot have a convectively unstable troposphere. This conclusion is obviously incorrect for the Earth's atmosphere, since in spite of possessing a primarily diatomic atmosphere, its troposphere is convectively unstable.

In order to modify this solution, we allow the scale height of the absorbing atmosphere to be different from that of the primary atmosphere. Equation (15b) is modified to

$$\tau = \tau^* e^{-z/Hg} \quad (17)$$

where we have used Hg, the scale height of the absorbing gas, rather than H, the scale height of the atmosphere. With this modification, (16) becomes

$$\frac{\Gamma_R}{\Gamma_D} = \frac{H}{Hg}\frac{C_p}{4R}\left(\frac{D\tau}{1 + D\tau}\right) \quad (18a)$$

For $\tau \gg 1$ and a primarily diatomic atmosphere, (18a) becomes

$$\frac{\Gamma_R}{\Gamma_D} = \frac{7}{8}\frac{H}{Hg} \quad (18b)$$

We see that an additional criterion involving the ratio of scale heights has entered into the stability. For Earth, the primary absorber is H_2O whose scale height is about 2 km, whereas $H \approx 8$ km such that $\Gamma_R/\Gamma_D \approx 3.5$. This is strongly unstable. Incidentally, the scale height of water vapor can be roughly accounted for by differentiating the Clausius-Clapeyron relation governing H_2O saturation vapor pressure (e_s) with respect to altitude. We find that e_s decreases with height with an approximate e-folding length of

$$H_{\text{vapor}} = \frac{R_v\langle T\rangle^2}{L\langle\Gamma\rangle} \quad (19)$$

R_v is the gas constant for water, $\langle T\rangle$ is the mean temperature of the given atmospheric column, L is the latent heat of evaporation for water, and $\langle\Gamma\rangle$ is the mean lapse rate. Equation (19) would approximate the scale height of vapor pressure (rather than saturation vapor pressure) if relative humidity were constant with altitude. This expression is only a crude approximation, since in general, relative humidity, T, and Γ are functions of altitude (and, in the case of Γ, water vapor as well from a moist lapse rate relation). However, using a mean temperature of 270 K and a mean lapse rate of 6.5 K km^{-1} gives $H_{\text{vapor}} = 2.1$ km. This is approximately what is observed.

For pure CO_2 atmospheres such as are found on Mars and Venus, $H/Hg = 1$, and $\Gamma_R/\Gamma_D = (C_p/4R)[D\tau/(1 + D\tau)]$. For Venus, τ is $\gg 1$, $C_p/R \approx 4.33$, and therefore $\Gamma_R/\Gamma_D \approx 1.1$ (slightly unstable troposphere). In summary, when we account for the difference in absorber scale heights, Schwarzschild's equations predict convectively unstable tropospheres for the Earth and Venus. The main remaining deficiency at this stage is that the solution predicts only a slight instability ($\gamma \approx 1$) for Venus, which has a very deep troposphere.

3.3. Effects of Strong Pressure-Broadened Lines

At this point we must make the distinction between optical depth scale height and concentration scale height. We have implicitly assumed, thus far, that the optical depth (τ) is independent of the total atmospheric pressure (recall (2a)), except where atmospheric pressure determines the partial pressure of the absorber (recall (15b)) through a mixing ratio relation. This assumption is valid for the following conditions: (1) optically thin atmosphere (τ is much less than 1), or (2) the primary line broadening mechanism is Doppler broadening, rather than collision broadening.

Neither of these two conditions hold for the lower atmospheres of Mars, Earth, or Venus. For these planetary atmospheres, the lines are broadened by collisional processes for which the absorption by a single line depends on the total atmospheric pressure as well as the absorber partial pressure. We will now consider the case in which the line center is optically thick, such that the transfer of radiation occurs through the line wings. This case is valid for the lower atmospheres of Mars, Earth, and Venus.

The optical depth for the far wings of pressure-broadened Lorentzian lines in such atmospheres can be expressed as

$$\tau(z) = K_B \int_z^\infty pP \, dz \tag{20}$$

P is the atmospheric broadening pressure, while p is the partial pressure of the absorber. K_B is the absorption proportionality coefficient for pressure-broadened lines (note that its units are different from those of K in (2a)). It is important to recognize that in the case we are considering, the optical depth of the line centers is extremely high and remains high at all pressures in the troposphere. Thus the vertical gradient in absorption by the line centers is very small in the troposphere compared with the gradient in absorption by the line wings. It is for this reason that most of the radiative transfer occurs in the line wings, allowing us to consider in our discussion optical depth in the line wings alone.

Recall that

$$p = p^* e^{-z/Hg} \tag{21a}$$

$$P = P^* e^{-z/H} \tag{21b}$$

where p^* and P^* are the surface pressures of the absorber and the whole atmosphere, respectively, H is the scale height of the primary atmosphere, and Hg is the absorber scale height. Combining (20), (21a), and (21b), we have

$$\tau(z) = K_B p^* P^* \int_z^\infty e^{-z/H_\tau} \, dz \tag{22a}$$

$$\frac{1}{H_\tau} = \frac{1}{H} + \frac{1}{Hg} \tag{22b}$$

H_τ is the effective scale height for the optical depth of the absorber. For a well-mixed gas such as CO_2, $Hg = H$, and the appropriate optical depth scale height is $H_\tau = H/2$, half the scale height of the primary atmosphere. Pressure broadening increases the fall-off of optical depth with height over that of absorber amount with height. In other words, the optical depth of an absorber in the strong-line limit is weighted by total atmospheric pressure squared and thus decreases with height more rapidly than concentration. In order to realistically represent planetary atmospheres such as Earth's, we must replace Hg in (17), (18a), and (18b) with H_τ from (22b):

$$\frac{\Gamma_R}{\Gamma_D} = \frac{H}{H_\tau} \frac{C_p}{4R} \left(\frac{D\tau}{1 + D\tau} \right) \tag{23a}$$

or, upon substitution of (22b),

$$\frac{\Gamma_R}{\Gamma_D} = \left(1 + \frac{H}{Hg} \right) \frac{C_p}{4R} \left(\frac{D\tau}{1 + D\tau} \right) \tag{23b}$$

Considering Venus, for which $H = Hg$ and $\tau \gg 1$, we have

$$\frac{\Gamma_R}{\Gamma_D} = 0.5 \frac{C_p}{R} \approx 2.17$$

For Earth, $H/Hg = 4$, and we have for $\tau \gg 1$,

$$\frac{\Gamma_R}{\Gamma_D} = 1.25 \frac{C_p}{R} \approx 4.38$$

Both cases are strongly unstable with respect to convection. Thus we can state: An optically thick atmosphere whose primary absorbing-emitting species is in the strong-line limit is absolutely convectively unstable, irrespective of whether the primary atmospheric constituent is monatomic, diatomic, or triatomic.

4. Comparison With Numerical Multispectral Models

The modified SGM significantly alters the conclusions of the SGM. We now examine how the results presented above compare with those from more accurate and detailed models.

First, the modified SGM shows that the temperature jump between the surface and the immediately overlying atmosphere vanishes inversely as the total optical depth weighted by the size of the spectral window. If we use (12) with a nonzero β and allow the optical depth to be high somewhere in the spectrum (as it is in the centers of the primary vibration-rotation bands of CO_2 and H_2O), we find that the discontinuity should be small. Indeed, this has been shown to be true for the model of *Manabe and Strickler* [1964], who state that the discontinuity is small due to very strong absorption near the line centers as well as upward radiation from the surface through the nearly transparent regions in the line wings and the water vapor window.

Second, the modified SGM concludes that, for an optically thick atmosphere, the equilibrium surface temperature for a given amount of solar radiation is determined by the size of the thermal window, and not by the optical depth in the thick part of the spectrum. For the case of Venus, CO_2 by itself, in spite of its high concentration, cannot explain the high surface temperature because it would leave too large a window. Opacity from other gases is required to fill the window. *Pollack* [1969] employed one of the most detailed spectral models of Venus to reach this same conclusion.

Third, the modified SGM predicts that the stability of the equilibrium atmospheric thermal structure depends on the ratio of the atmospheric pressure scale height to the absorber optical depth scale height. In order to test this prediction, we converted the radiation code in the National Center for Atmospheric Research (NCAR) Community Climate Model, version 2 (CCM2) to a radiative equilibrium model using a simple time-stepping method [*Manabe and Strickler*, 1964]. We modified the temperature profile at each interval with the sum of the longwave and solar heating rates multiplied by a time step of 4 hours. We ran the model for 7000 time steps for each scenario, though the model was close to equilibrium after approximately 2000.

In order to mimic the modified SGM, which allows for only one optically active constituent, we deactivate the CO_2 effect by reducing its concentration to 10^{-4} of its present value. We also remove the effect of O_3 and the solar absorption by H_2O, CO_2, and O_3. In all cases the value of the H_2O mixing ratio at the surface is 4.0 g kg^{-1}. Water vapor is the equivalent of the single absorber in the strong-line limit as in the previous section. We allow H_τ to vary by

Table 1. Comparison Between CCM2-Calculated Lapse Rate and Modified SGM-Predicted Lapse Rate for Different Values of H/H_τ

H/H_τ	Average Lapse Rate Between 1000.5 and 974.8 mbar From CCM2, K km^{-1}	Modified SGM Lapse Rate, K km^{-1}
2.50	15.67	21.44
2.00	14.39	17.15
1.67	13.02	14.32
1.43	11.49	12.26
1.25	9.81	10.72
1.11	7.85	9.52

For the modified SGM results, we have used Γ_R from (23a), assuming Γ_D = 9.8 K km^{-1} and $\tau \gg 1$.

modifying the variation of water vapor mixing ratio with height. Table 1 shows a comparison between the average lapse rate in the second model level of the CCM2, and the lapse rate predicted by the modified SGM for the same ratio H/H_τ. Recall that $H/H_\tau = 2$ corresponds to a well-mixed gas. The two sets of numbers describe a similar trend, indicating a similarity in behavior between the modified SGM and the CCM2 radiative equilibrium model.

5. Concluding Remarks

The simple, heuristic modifications we propose for Schwarzschild's equations rectify a number of flaws in the SGM predictions for planetary atmospheres. The solutions of the SGM and the modified SGM are presented for comparison and reference in Table 2.

The modified SGM is a powerful yet simple analytical tool for conceptualizing many fundamental aspects of radiative transfer and thermal structure in planetary atmospheres. Table 2 provides a summary of the model solutions for both the original SGM, as well as the modified SGM. The simple modifications allow the model to approach reality qualitatively much more closely than the original SGM. Such tools are important in climate studies because they allow one to grasp the essence of a complex subject all at once.

However, we point out that simple models such as the present one cannot be depended upon for quantitative accuracy; hence they should be used with extreme care to avoid misinterpretations and erroneous conclusions. For example, it is well known that water vapor and CO_2 absorption do not follow semigrey gas models. It is for this reason that we focus on the optically thick limit for which the influence of the spectral details is expected to be minimal.

The principal predictions of the modified SGM lead to a number of corollaries. We point out the most important of these to perhaps motivate detailed studies with more accurate radiation and climate models. Such studies could determine if these corollaries are valid under more general conditions, such as radiative-convective equilibrium. In addition, it is worthwhile to note that the above results apply only when most of the solar absorption happens at the surface.

First, a diatomic atmosphere with an optically thick absorber will have a convectively unstable troposphere if one (or both) of the following conditions are satisfied:

1. The scale height of the radiatively active gas is smaller than that of the atmosphere.
2. The absorption lines are broadened by collisional processes (Lorentzian lines).

According to this corollary, the Earth's atmosphere (which is primarily diatomic) is convectively unstable because both conditions are satisfied for water vapor. Even without water vapor, Earth would be convectively unstable through condition 2, provided the CO_2 concentration was high enough. However, the high degree of instability of the present-day atmosphere is due to the scale height of water vapor partial pressure being much smaller than the atmospheric pressure scale height (i.e., $H/Hg \approx 4$).

Second, in a triatomic atmosphere with a uniformly mixed, optically thick gas (e.g., Venus), the radiative equilibrium structure is always unstable, but the departure from

Table 2. Solutions to SGM and Modified SGM

Quantity	SGM	Modified SGM
$T(\tau)$	$\left(\frac{f_0}{\sigma}\right)^{1/4}\left(\frac{1+D\tau}{2}\right)^{1/4}$	$\left(\frac{f_0}{\sigma}\right)^{1/4}\left(\frac{1+D\tau}{2+\beta D\tau^*}\right)^{1/4}$
T_g	$\left(\frac{f_0}{\sigma}\right)^{1/4}\left(\frac{2+D\tau^*}{2}\right)^{1/4}$	$\left(\frac{f_0}{\sigma}\right)^{1/4}\left(\frac{2+D\tau^*}{2+\beta D\tau^*}\right)^{1/4}$
$\sigma T_g^4 - \sigma T(\tau^*)^4$	$\frac{f_0}{2}$	$f_0\frac{1}{(2+\beta D\tau^*)}$
T_g for $\tau^* \gg 1$	$\left(\frac{f_0}{\sigma}\right)^{1/4}\left(\frac{D\tau^*}{2}\right)^{1/4}$	$\left(\frac{f_0}{\sigma\beta}\right)^{1/4}$
$-\frac{dT(\tau)}{dz}$	$\frac{g}{4R}\left(\frac{D\tau}{1+D\tau}\right)$	$\frac{g}{4R}\frac{H}{H_\tau}\left(\frac{D\tau}{1+D\tau}\right)$
$-\frac{dT(\tau)}{dz}$ for $\tau^* \gg 1$	$\frac{g}{4R}$	$\frac{g}{4R}\frac{H}{H_\tau}$

Here, τ, optical depth (2a); τ^*, surface optical depth (2b); f_0, incoming solar radiation (4); β, energy-weighted spectral window (8); H, atmospheric pressure scale height (15c); H_τ, optical depth scale height (22b).

a neutral lapse rate is twice as large for pressure-broadened lines (due to condition 2 above).

Third, for an optically thick atmosphere with transparent spectral regions, the surface temperature and its response to external parameters are strongly governed by the energy-weighted width of the spectral window. For the Earth's atmosphere, the 8- to 12-μm window (rather than the details of the optically thick pure rotation band) governs the surface temperature and its response. For example, with an increase in surface temperature, if the water vapor content increases, the width of the window decreases and leads to a positive feedback.

Appendix: Modified SGM With Solar Absorption in the Atmosphere

For completeness, we extend the semigrey equations to include absorption of solar radiation by the atmosphere. For any τ, the net longwave flux is balanced by a net solar flux:

$$F = f_0 e^{-\tau/h} \qquad \text{(A1)}$$

where h is a dimensionless quantity representing the optical depth at which the net (down − up) solar radiation reaches $1/e$ of its top-of-the-atmosphere value. Here we have used τ to again represent the optical depth in the thermal part of the spectrum, and as regards the solar radiation, to provide a measure of vertical position in the atmosphere. In a manner analogous to (6a) and (6b) or (11a) and (11b), we can write the temperatures of the atmosphere and surface as follows:

$$\sigma T^4 = f_0 \left\{ \left[\frac{(1 + Dh)[e^{-\tau/h}(1 - Dh) + Dh]}{2Dh(1 - \beta)} \right] - \left[\frac{\beta(1 + D\tau)[e^{-\tau^*/h}(1 - Dh) + (1 + Dh)]}{2(1 - \beta)(2 + \beta D\tau^*)} \right] \right\} \qquad \text{(A2a)}$$

$$\sigma T_g^4 = f_0 \left[\frac{1 + e^{-\tau^*/h} + Dh(1 - e^{-\tau^*/h})}{2 + \beta D\tau^*} \right] \qquad \text{(A2b)}$$

Using $H = RT(\tau)/g$ and $\Gamma_d = g/C_p$, we can write the lapse rate as follows:

$$\Gamma = \Gamma_D \frac{H}{Hg} \frac{C_p}{4R} \tau \left\{ \left[\frac{-E(1 + Dh)}{2Dh^2(1 - \beta)} - \frac{\beta D[E^* + (1 + Dh)]}{2(1 - \beta)(2 + \beta D\tau^*)} \right] \cdot \left[\frac{(1 + Dh)(E + Dh)}{2Dh(1 - \beta)} - \frac{\beta(1 + D\tau)[E^* + (1 + Dh)]}{2(1 - \beta)(2 + \beta D\tau^*)} \right]^{-1} \right\} \qquad \text{(A3)}$$

where $E = e^{-\tau/h}(1 - Dh)$ and $E^* = e^{-\tau^*/h}(1 - Dh)$. For the case of pressure-broadened thermal absorption, replace Hg with H_τ as described in section 3.3.

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References

Goody, R. M., and Y. L. Yung, *Atmospheric Radiation: Theoretical Basis*, 2nd ed., 519 pp., Oxford University Press, New York, 1989.

Houghton, J. T., *The Physics of Atmospheres*, 2nd ed., 271 pp., Cambridge University Press, New York, 1986.

Manabe, S., and R. F. Strickler, Thermal equilibrium of the atmosphere with a convective adjustment, *J. Atmos. Sci.*, *21*, 361–385, 1964.

Manabe, S., and R. T. Wetherald, Thermal equilibrium of the atmosphere with a given distribution of relative humidity, *J. Atmos. Sci.*, *24*(3), 241–259, 1967.

Pollack, J. B., A nongray CO_2-H_2O greenhouse model of Venus, *Icarus*, *10*, 314–341, 1969.

Ramanathan, V., and J. A. Coakley Jr., Climate modeling through radiative-convective models, *Rev. Geophys.*, *16*(4), 465–489, 1978.

Schwarzschild, K., On the equilibrium of the sun's atmosphere (1906), in *Selected Papers on the Transfer of Radiation*, edited by D. H. Menzel, pp. 25–34, Dover, Mineola, N.Y., 1966.

Simpson, G. C., Some studies in terrestrial radiation, *Mem. R. Meteorol. Soc.*, *II*(16), 69–95, 1927.

Simpson, G. C., Further studies of terrestrial radiation, *Mem. R. Meteorol. Soc.*, *III*(21), 1–27, 1928.

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